Marin Math Circle — Two is Better Than Infinity

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1 The **triangular numbers** are the sums of consecutive integers, starting with 1. The first few triangular numbers are

$$1, \quad 1+2=3, \quad 1+2+3=6, \quad 1+2+3+4=10, \ldots.$$

Prove that if T is a triangular number, then $8T+1$ is a perfect square. Is the converse true? In other words, if S is a perfect square, must there be a triangular number T such that $8T+1=S$?

2 Can every positive integer be written as a sum of two or more consecutive positive integers? If so, prove it. If not, show (with proof) which integers can and which cannot.

3 If 127 people play in a singles tennis tournament, prove that at the end of tournament, the number of people who have played an odd number of games is even.

4 Let $a_1, a_2, \ldots, a_n$ represent an arbitrary arrangement of the numbers $1, 2, 3, \ldots, n$. Prove that, if n is odd, the product

$$(a_1-1)(a_2-2)(a_3-3)\cdots(a_n-n)$$

is an even number.

5 What are the maximum number of "Friday the 13ths" that can occur in a normal 365-day year? What are the minimum number that must occur? (Recall that April, June, September and November each have 30 days, February has 28 days, and all the other months have 31 days.)

6 Three frogs are placed on three vertices of a square. Every minute, one frog leaps over another frog, in such a way that the "leapee" is at the midpoint of the line segment whose endpoints are the starting and ending position of the "leaper." Will a frog ever occupy the vertex of the square that was originally unoccupied?

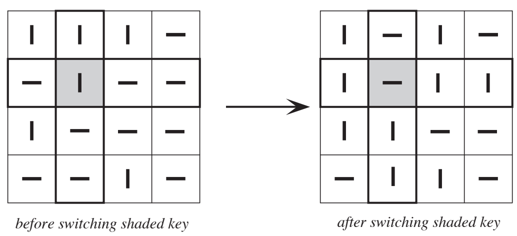
7 *Closed Paths*. Consider the following two-person game: You start with an $n \times m$ grid of graph paper. Players take turns coloring red one previously uncolored unit edge of the grid (including the boundary). The player who creates the first closed red path loses. Is there a winning strategy for one of the players?

8 Determine the largest number which is the product of positive integers whose sum is 2009.

9 An evil wizard has imprisoned 64 math circle participants. The wizard announces, "Tomorrow I will have you stand in a line, and I will put a hat on each of your heads. The hat will be colored either white or black. You will be able to see the hats of everyone in front of you, but you will not be able to see your hat or the hats of the people behind you. I will begin by asking the person at the back of the line to guess his or her hat color. If the guess is correct, that person will get a cookie. If the guess is wrong, that person will be killed in a painful way. Then I will ask the next person in line, and so on. You are only allowed to say the single word 'black' or 'white' when it is your turn to speak, and otherwise you are not allowed to communicate with each other while you are standing in line. Although you will not be able to see the people behind you, you will know (by hearing) if they have answered correctly or not."

The prisoners are allowed to chat for a few minutes before their ordeal begins. What is the largest number of prisoners that can be *guaranteed* to survive?

10 (Bay Area Mathematical Olympiad 1999) A lock has 16 keys arranged in a 4×4 array, each key oriented either horizontally or vertically. In order to open it, all the keys must be vertically oriented. When a key is switched to another position, all the other keys in the same row and column automatically switch their positions too (see diagram). Show that no matter what the starting positions are, it is always possible to open this lock. (Only one key at a time can be switched.)



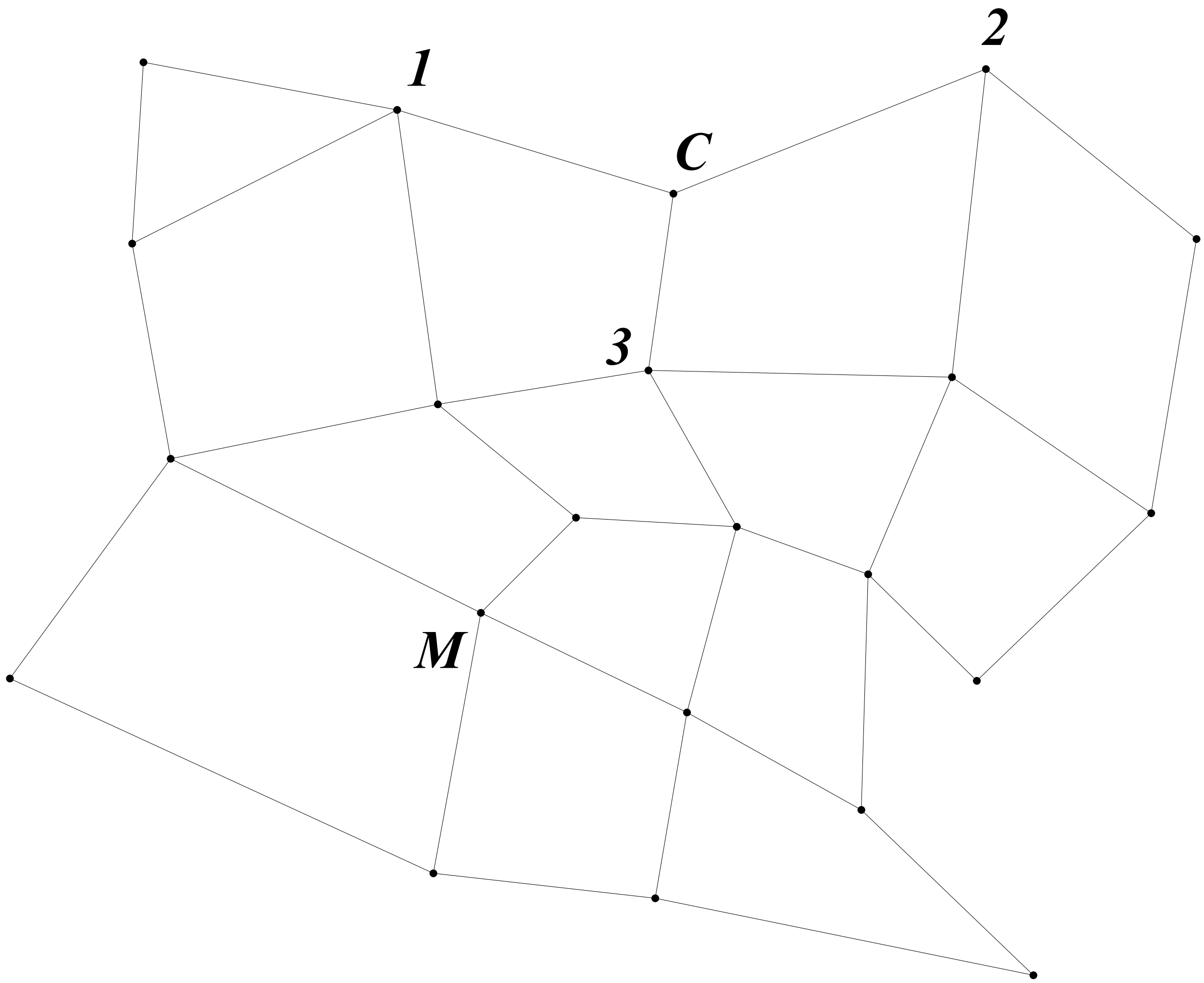
11 (a) A woman possesses five stones. She claims that with a simple two-pan balance, using some combination of her stones, she can balance any rock you give her that weighs a whole number of pounds up to 31 pounds. What are the weights of her five stones?

(b) A second woman possesses a different set of five stones. She claims that, with a simple two-pan balance she can determine the weight of any rock you give her that weighs a whole number of pounds up to 121 pounds. What are the weights of her five stones? (Note: This woman does not claim that she can balance the rock with a combination of her stones, only that she can determine what its weight must be.)

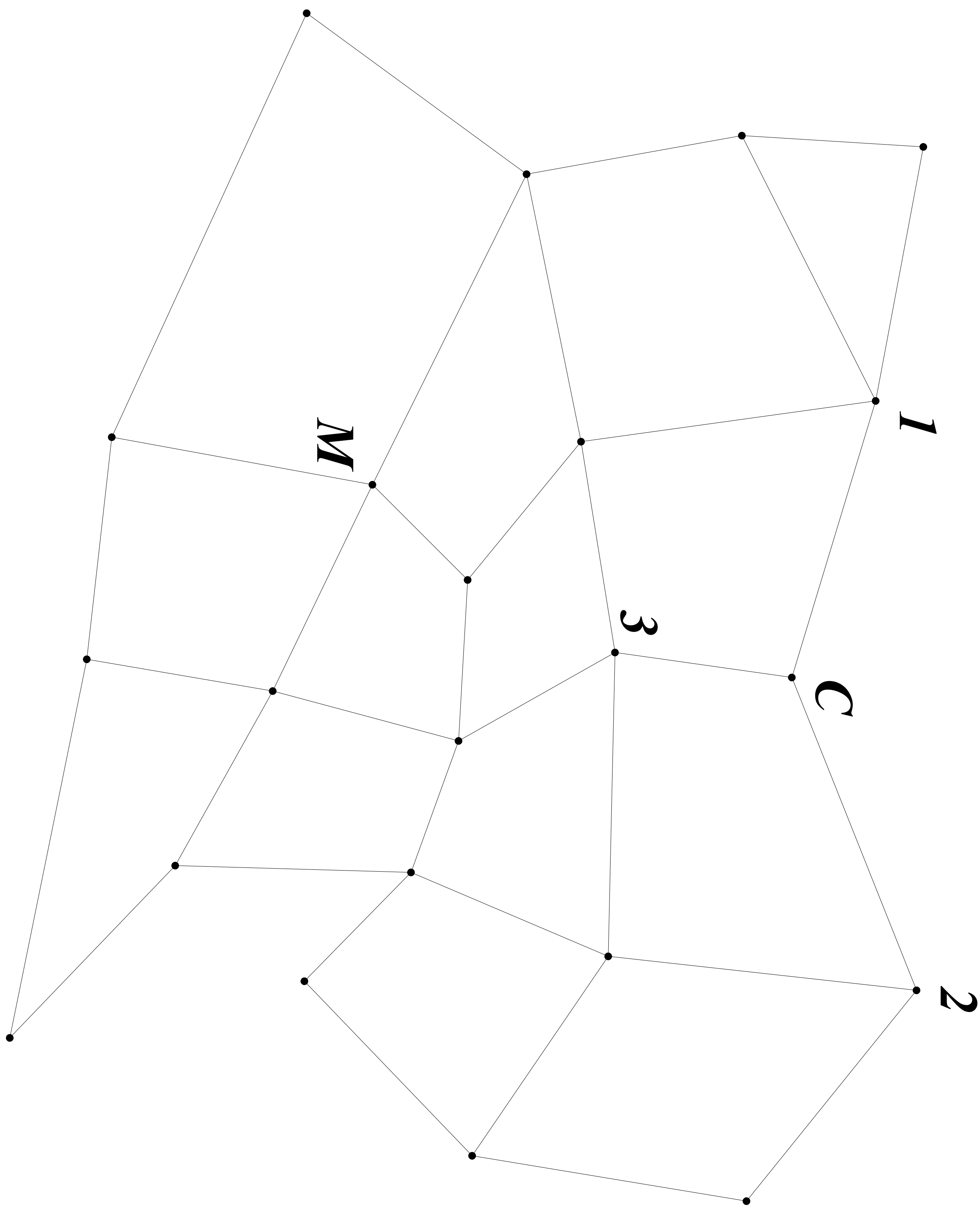
(c) A third woman has yet another set of five stones. Using a simple two-pan balance she can determine the weight of any rock you give her weighing a whole number of pounds up to 242 pounds! What are the weights of her five stones?

12 Twenty-three people, each with integral weight, decide to play football, separating into two teams of eleven people, plus a referee. To keep things fair, the teams chosen must have equal *total* weight. It turns out that no matter who is chosen to be the referee, this can always be done. Must these 23 people all have the same weight?

13 *Cat and Mouse*. A very polite cat chases an equally polite mouse. They take turns moving on the grid depicted below.



Initially, the cat is at the point labeled C; the mouse is at M. The cat goes first, and can move to any neighboring point connected to it by a single edge. Thus the cat can go to points 1, 2, or 3, but no others, on its first turn. The cat wins if it can reach the mouse in 15 or fewer moves. Can the cat win?


M
1
3
C
2