

About the Marin Math Circle – Advanced Group
A description and notes on a session
by Jon Sigerman, a parent and organizer for the program

OVERVIEW

The Advanced Marin Math Circle at DUC offers engaging, interactive topic roundtables, each led by a distinguished mathematician. The topic de jure covers one or two evening sessions. The sessions are conducted in a format that affords all participants the opportunity to ponder, discuss and collaboratively solve challenging stepping stone problems leading up to an interesting or even beautiful mathematical result.

High-school participants in the advanced circle oftentimes need to juggle the demands of honors courses, sports and other extra-curricular activities. As a result no participant is expected be present for all topics. Because topics are relatively independent, participants can derive tremendous benefits and enjoyment from the advanced circle without having to attend every session.

An example of what advanced math circle is about is presented below in a recap of the session proceedings for the two session 2011-2012 topic moderated by Dr. Paul Zeitz, Professor of Mathematics, University of San Francisco, and author of the book “ The Art and Craft of Problem Solving”; Wiley Press.

EXAMPLE TOPIC:

“How, What and Why” of problem-solving; in particular, the fundamental problem solving techniques of:

(1) wishful thinking, and

(2) pictures , number patterns, investigations

(Marin Math Circle, September 21, 2011; see

<http://www.marinmathcircle.org>)

Notes by parent and math circle organizer Jon Sigerman.

Summary of proceedings:

This topic session, like the commencement of most intermediate and advanced math circle sessions, began with a hand-out of foundational problems framing the topic, and forming the itinerary for exploration. Appendix 1 at the end of this write-up contains excerpts from the topic problem set handout. [The full session hand-out can be downloaded from the “Schedule” link at <http://www.marinmathcircle.org>].

1) wishful thinking problem solving technique

Participants were asked to take a few minutes to peruse the first 5 problems of Appendix 1, which follows. (You should do the same now). The question then was posed: Can you see why these 5 problems are “Five Versions of the Same Problem”? Answer: “They all involve the technique of wishful thinking”.

“Wishful thinking” as applied to problem solving technique, means, “I wish the given problem looked like THIS; because I can solve THIS easily.” In other words, the thought process is: See if the given problem can be re-jiggered, simplified, or partitioned to look like a simpler, imminently solvable (or previously solved) problem. If so, then appropriately adjust the wished for problem to comply with the constraints of the given problem.

The group, in collaboration with Dr. Zeitz , worked through problems (a) and (b) to illustrate the technique “wishful thinking”.

Problem (b): Factor the fourth order polynomial expression x^4+x^2+1 .

Apply the wishful thinking thought process:

Factoring x^4+x^2+1 is tough; I wish that I had been given the simpler task of factoring x^4+2x^2+1 , which is readily be seen to be $(x^2 + 1)^2$. Having identified a related simpler task begs the next question: Can x^4+x^2+1 , indeed, be re-jiggered so I can use the solution for $x^4 + 2x^2 + 1$ in an expression that is easily factored, yet remains equivalent to x^4+x^2+1 ?

Yes, I can: $x^4 + 2x^2 + 1 - x^2$.

Since the latter expression fits the form $a^2 - b^2 = (a-b)(a+b)$, it is easy to see $x^4 + 2x^2 + 1 - x^2 = (x^2 + 1 - x)(x^2 + 1 + x)$.

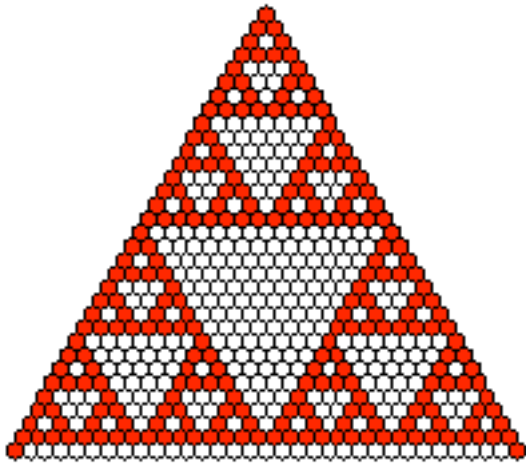
Appendix 2 contains hints and solutions for the remaining Appendix 1 Problems.

(2) Problem solving technique of creating pictures, spotting patterns, investigating the leads arising from the pictures and patterns

Upon completion of the **wishful thinking** segment Dr. Zeitz ratcheted up the difficulty level in the next segment: (2) pictures, number patterns, investigations. Binary- fission reproducing cannibals, Fibonacci series, and Pascal's Triangle were featured in problems for this segment. The continuation of our summary of proceedings will focus on investigation into parity patterns found in Pascal's Triangle:

Problem: As Pascal's triangle gets very large, e.g. a billion trillion rows, what if anything can be said about the relative frequency of even and odd numbers?

The picture below converting Pascal Triangle numerical entries into odd (red) and (even) elements is worth a thousand words.



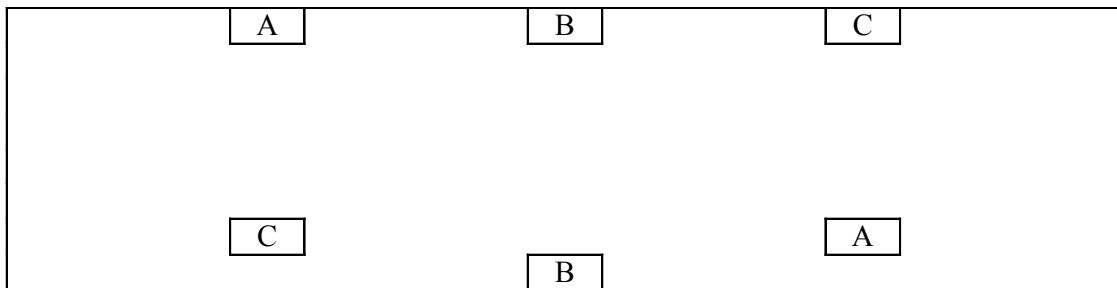
The detailed analysis leading to the solution appears in Appendix 3. As a preview to Appendix 3, note the last row of the above image. It follows a row of all odd numbers (red hexagons), and starts off with a red-hexagon (i.e. the number 1) at each end, bordering a bunch of zeros. Note also the vertex of Pascal's Triangle consists of a red hexagon surrounded by white space. As your intuition should tell you the Pascal's Triangle number generating function will generate two perfect replicas or clones of the triangle generated by the first 31 rows, starting with row zero, the vertex row of Pascal's Triangle consisting of a single red hexagon at the top. The all odd row (2^5 or 32 red hexagons) of the two clones will bump into each other 32 rows down Pascal's Triangle (i.e row $2^6 - 1$, or row 63). Equally important, you note that the two clones surround a desert of even numbers or white hexagons, caused by the fact that when the number generating function in Pascal's triangle goes to work, when an odd is added to a neighboring odd in the row above, an even number is generated; likewise, when an even is added to a neighboring even in the row above, an even number is generated. So, you conjecture that the number of red hexagons or odd numbers in Pascal's triangle increases by a factor of 3 as the number of new rows added to the triangle increase by a power of 2. When you translate all of the above into counting algebra, you see the proportion of odd numbers (red hexagons) to total hexagons converges to zero as get to very, very

large row numbers in Pascal's triangle. The details are found in Appendix 3, which follows.

Appendix 1

1 Five Versions of the Same Problem.

(a) Consider the following diagram. Can you connect each small box on the top with its same-letter mate on the bottom with paths that do not cross one another, nor leave the boundaries of the large box?



(b) Factor x^4+x^2+1 .

(c) You are in the downstairs lobby of a house. There are 3 switches, all in the "off" position. Upstairs, there is a room with a lightbulb that is turned off. One and only one of the three switches controls the bulb. You want to discover which switch controls the bulb, but you are only allowed to go upstairs once. How do you do it? (No fancy strings, telescopes, etc. allowed. You cannot see the upstairs room from downstairs. The lightbulb is a standard 100-watt bulb.)

(d) Pills. For 10 days, you must take one A pill and one B pill at noon. Otherwise, you die. If you take too much or too little medicine, you will die. The pills are indistinguishable! All goes well until day 3. On this day, you shake one A and TWO B pills into your palm. Can you survive? If so, HOW?

(e) You are locked in a $50 \times 50 \times 50$ -foot room which sits on 100-foot stilts. There is an open window at the corner of the room, near the floor, with a strong hook cemented into the floor by the window. So if you had a 100-foot rope, you could tie one end to the hook, and climb down the rope to freedom. (The stilts are not accessible from the window.) There are two 50-foot lengths of rope, each cemented into the ceiling, about 1 foot apart, near the center of the ceiling. You are a strong, agile rope climber, good at tying knots, and you have a sharp knife. You have no other tools (not even clothes). The rope is strong enough to hold your weight, but not if it is cut lengthwise. You can survive a fall of no more than 10 feet. How do you get out alive?

Appendix 2:

Problem (a):

Hint 1:

Wish I wish I could move Block A to block C's spot, and vice versa, then connect the blocks.

Hint 2: adjustment—After re-jiggering the blocks so you have your wish as above, connect the 3 blocks with stretchable different colored cords so there are no overlaps of cord.

Solution: Move block A back to its original location pushing the stretchable cords with the side of block A so the colored cords don't cross each other. Move Block back to its original position, again pushing the stretchable cord so there is no overlap.

Problem (c):

Hint 1: I wish that a light bulb had more than two states, that is more than just lit or not lit.

Hint 2: What are the properties of a 100 watt incandescent that has been left on for an extended period of time? What are the properties of an incandescent bulb that has just been turned on.

Solution: Turn switch 1 to on position—hang out downstairs for awhile long enough for a lighted bulb to get warm. Then turn switch 2 to on position, and run like crazy upstairs. You have left switch 3 off. When you get in the room, immediately look to see if the bulb is on; if on feel it to see if it is hot. If on and hot, switch 1 is connected. If on and room temp or so, switch 2 is connected. If off switch 3 is connected.

Problem (d):

Hint 1: I wish I had apothecary tools, and a pill-sized spoon.

Hint 2: If I didn't have apothecary tools, I wish I had a pill slicer that was like a miniature perfect pie slicer with section fractions that can be pre-set.

Solution:

Since it is noon on Day 3, put the 3 pills aside into one cup. Then, carefully this time, take one A pill from bottle A, and take your 1 B pill from bottle B. You have survived to live another day. Then use the pill slicer to slice, each of the three pills you're your screw-up into thirds, keeping together each $\frac{1}{3}$ from the same pill. (e.g. put each $\frac{1}{3}$ slice from the same pill into a separate cup). Take a $\frac{1}{3}$ segment from each cup, and put each segment into three more cups marked cups with roman

numbers I, II, and III respectively. Each I-III cup will contain the equivalent of a single pill with $\frac{2}{3}$ dose of B and $\frac{1}{3}$ dose of A. For days 4-7 take a pill from the A bottle and a pill from the B bottle. On the morning of day 8, you have 2 pills left in bottle A, and 1 pill left in bottle B. Before noon, use the pill slicer again to cut the two pill A pills into 6 perfect $\frac{1}{3}$ pill slices. Put those in a cup labeled A. Cut the remaining B pill into $\frac{1}{3}$ slices; put the three slices into a cup labeled B. At noon take 3 slices from cup I, 2 slices from cup A, and 1 slice from cup B, and you are guaranteed to have for your noon survival dose of A and B for day 8 because cup I gives you $2 \times \frac{1}{3}$ pill B slices plus $1 \times \frac{1}{3}$ pill A slice. The 2 slices from cup A gives you $2 \times \frac{1}{3}$ pill A slices. The 1 slice from cup B gives you $1 \times \frac{1}{3}$ cup B slice. Do the same for days 9 and 10.

Problem (e):

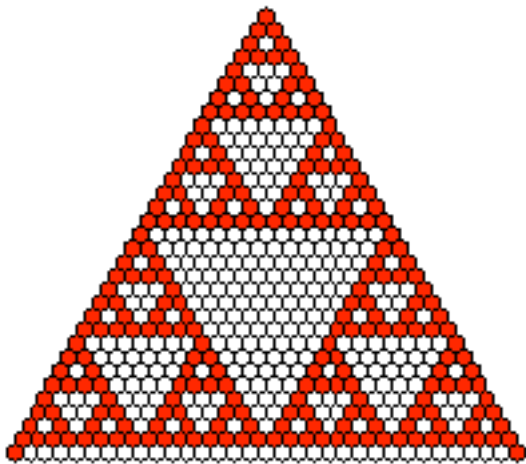
Hint 1: I wish there was a single 100 foot length of rope hanging from a loop in the ceiling.

Hint 2: I wish I could slide the rope through the loop even when the rope has a knot in it.

Solution: Invoke the gymnast's ability to climb to the top where the ropes are cemented, and tie the two ropes into a square knot so a loop with both ropes hanging down from the knot ends is formed. The gymnast remains at the top with his knife, hanging by his knees from the loop formed with the knot and cement junctions, clips off each of the ropes so he has 2 long segments, slip the end of one segment through the loop tie it the other cut segment, then shimmy down the hanging rope, so that neither slides to far off of the loop , until the gymnast gets less than 10 feet from the floor of the building. The rope is pulled through the loop, so it is freed from the ceiling to be used for the climb to the ground.

This is the point where Dr. Zeitz asked the rhetorical question “How do you approach an investigation of the problem?, then answered, in effect, first get a sense of the beast. Observe some of the patterns made by rows, columns, diagonals, and staircases. The participants proceed to articulate various patterns.

Turning back to the problem at hand, Dr. Zeitz suggested that since problem at hand involves only odd and even numbers, simplify our pictures to evens and odds. Some of the participants drew Pascal Triangles pictures consisting exclusively of 1s and 0s. Others converted the classic Pascal’s Triangle pictured above to just red hexagons for odd numbers, and white hexagons for even numbers. See the picture below. An amalgamated mental model of both pictures formed the foundation for the investigation, with a red hexagon representing the number 1 and a white hexagon representing 0. A number (hexagon color) in a row under construction was determined via addition of two neighboring reference row elements in base 2, modulo 2.



The next step after everyone used only a colored-coded picture to make some conjecture about the relative frequency of red and white area was to arrive at a nomenclature for the problem components. The objective is to facilitate algebraic investigation once visual and tabular investigation bogs down.

Our first nomenclature convention: let n = the row number of the triangle. In the introductory picture defining Pascal’s triangle, 0 was designated to represent the row number occupied by the lone 1 at the vertex. So for the second row (1,1), $n = 1$; $n = 2$ for (1,2,1); $n = 3$ for (1,3,3,1); and so on. By our nomenclature $n = 32$ for the last row in the above picture, the row with two red ends bordering a strip of 31 white hexagons.

Next , observe row $n = 31$ consisting of 32 red hexagons (i.e. 2^5 odd numbers). Row 31 forms the base for the biggest triangle in our picture whose base has all red hexagons. The second biggest triangle whose base (row 15) is formed with all red

hexagons contains 16 (i.e. 2^4) red hexagons. You should see a pattern, once you realize the “biggest” all red-based triangle appears to be formed by double cloning the top triangle formed by rows 0 through 15, and then stacking the cloned triangle on the vertices of its two clones. The two bases of the two clones line up to form row $n = 31$, the all red base of the biggest triangle. You also see that the three triplets so stacked, surround a triangular desert of white. You then conjecture that the number of red hexagons in the “biggest” triangle formed by rows 0 through 31 contains three times as many red hexagons as in the “second biggest” triangle formed by row 0 through 15.

An outline of the path to the solution, and the solution follows. Again, the more in depth discussion is found in the footnote.

Let $Y_K =$ the number of odds in Pascal’s triangle up to row $n = 2^K - 1$.

Then $Y_K = 3^K$ because of a recursive pattern of three identical triplet P_{K-1} triangles surrounding a desert of 0s comprises triangle P_K . P_{K-1} is the triangle of numbers which is made up of all rows of Pascal’s Triangle from row 0 to row $n = 2^{K-1} - 1$ (row $n = 2^k - 1$ composed of all odd numbers). It has been proven by induction that P_{K-1} contains 2^{K-1} odd numbers in row $n = 2^{K-1} - 1$. As such row $n = 2^{K-1}$ is composed of $2^{K-1} - 1$ even numbers together with two odd numbers at the extremes by virtue of how Pascal’s Triangle is generated. These two odd extremes form the seeds for generating two symmetrical clones of P_{K-1} on each side of the axis of symmetry of Pascal’s Triangle. Also, the row $n = 2^{K-1}$ even numbers which are symmetric about the axis of symmetry of Pascal’s Triangle generate strips of even numbers between the symmetrical clones above.

These two clones of P_{K-1} side by side surrounding the desert of even numbers form the next $2^{K-1} - 1$ rows of Pascal’s triangle; that is rows 2^{K-1} through $2^K - 1$. As such $Y_K = 3 \times Y_{K-1} = 3 \times 3^{K-1}$

Let $S_K =$ the total number of elements in Pascal’s triangle up to row $n = 2^k - 1$ (triangle P_k) which is equivalent to the total number of elements in our home grown stacked construction triangle T_K .

Then $S_K = (2^K)(2^k + 1) / 2$. Since the base of T_K has 2^K elements the total number of elements in T_K is $1 + 2 + 3 + 4 + 5 + \dots + 2^K$. [Dr. Zeitz gave a simple intuitive proof of the fact that the sum of successive positive integers from 1 to $n = n(n+1)/2$.]

So the proportion of odd elements to total elements in T_K $Y_K / S_K = 3^K$ divided by $(2^K)(2^k + 1) / 2 = 2 \times 3^K$ divided by $(2^K)(2^k + 1) = 2 \times 3^K$ divided by $4^K + 2^K$

$2 \times 3^K / 4^K + 2^K$ is less than $2 \times (3/4)^K$, so as K goes to infinity $2 \times (3/4)^K$ goes to 0, which means Y_K / S_K goes to zero. So, the answer to the problem is that in the limit the proportion of odd entries of total entries in Pascal’s triangle converges to 0.