Marin Math Circle
HOW, WHAT, AND WHY

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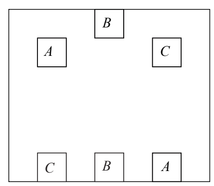
1 *Five Versions of the Same Problem.*

(a) Consider the following diagram. Can you connect each small box on the top with its same-letter mate on the bottom with paths that do not cross one another, nor leave the boundaries of the large box?



(b) Factor $x^4 + x^2 + 1$.

(c) You are in the downstairs lobby of a house. There are 3 switches, all in the "off" position. Upstairs, there is a room with a lightbulb that is turned off. One and only one of the three switches controls the bulb. You want to discover which switch controls the bulb, but you are only allowed to go upstairs once. How do you do it? (No fancy strings, telescopes, etc. allowed. You cannot see the upstairs room from downstairs. The lightbulb is a standard 100-watt bulb.)

(d) *Pills*. For 10 days, you must take one A pill and one B pill at noon. Otherwise, you die. If you take too much or too little medicine, you will die. The pills are indistinguishable! All goes well until day 3. On this day, you shake one A and TWO B pills into your palm.

Can you survive? If so, HOW?

(e) You are locked in a $50 \times 50 \times 50$-foot room which sits on 100- foot stilts. There is an open window at the corner of the room, near the floor, with a strong hook cemented into the floor by the window. So if you had a 100-foot rope, you could tie one end to the hook, and climb down the rope to freedom. (The stilts are not accessible from the window.) There are two 50-foot lengths of rope, each cemented into the ceiling, about 1 foot apart, near the center of the ceiling. You are a strong, agile rope climber, good at tying knots, and you have a sharp knife. You have no other tools (not even clothes). The rope is strong enough to hold your weight, but not if it is cut lengthwise. You can survive a fall of no more than 10 feet. How do you get out alive?

2 *Pictures, Numbers, and Investigation.*

(a) The **triangular numbers** are the sums of consecutive integers, starting with 1. The first few triangular numbers are

$$1, \quad 1+2=3, \quad 1+2+3=6, \quad 1+2+3+4=10, \ldots.$$

Prove that if T is a triangular number, then $8T+1$ is a perfect square. Is the converse true? In other words, if S is a perfect square, must there be a triangular number T such that $8T+1=S$?

(b) A number is called **trapezoidal** if it can be expressed as a sum of two or more consecutive positive integers.

  (a) Why are they called "trapezoidal?"

  (b) How do they relate to "triangular" numbers?

  (c) What numbers are trapezoidal?

(d) The **Fibonacci numbers** f_n are defined by $f_0 = 0, f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n > 1$. For example, $f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8, f_7 = 13, f_8 = 21$. Find patterns in this sequence. In particular, investigate parity (odd or even), divisibility by 3, divisibility by 5, perfect squares, etc.

(e) Here are the first few rows of **Pascal's Triangle**.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1,
\end{array}$$

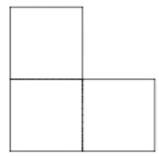
where the elements of each row are the sums of pairs of adjacent elements of the prior row. For example, $10 = 4+6$. The next row in the triangle will be

$$1, 6, 15, 20, 15, 6, 1.$$

There are many interesting patterns in Pascal's Triangle. Discover as many patterns and relationships as you can, and prove as much as possible. In particular, can you somehow extract the Fibonacci numbers (see next problem) from Pascal's Triangle (or vice versa)? Another question: is there a pattern or rule for the **parity** (evenness or oddness) of the elements of Pascal's Triangle?

(f) Determine the largest number which is the product of positive integers whose sum is 2005.

(g) An “ell” is an L-shaped tile made from three 1×1 squares (see picture). For what positive integers a, b is it possible to completely tile an $a \times b$ rectangle only using ells? For example, it is clear that you can tile a 2×3 rectangle with ells, but (draw a picture) you cannot tile a 3×3 with ells.



3 *Miscellaneous Fun Problems.*

(a) *Mixtures*. Bottle A contains a quart of milk and bottle B contains a quart of black coffee. Pour a small amount from B into A, mix well, and then pour back from A into B until both bottles again each contain a quart of liquid. What is the relationship between the fraction of coffee in A and the fraction of milk in B?

(b) *Weighing with stones and a scale.*

1. A woman possesses five stones. She claims that with a simple two-pan balance, using some combination of her stones, she can balance any rock you give her that weighs a whole number of pounds up to 31 pounds. What are the weights of her five stones?
2. A second woman possesses a different set of five stones. She claims that, with a simple two-pan balance she can determine the weight of any rock you give her that weighs a whole number of pounds up to 121 pounds. What are the weights of her five stones? (Note: This woman does not claim that she can balance the rock with a combination of her stones, only that she can determine what its weight must be.)
3. A third woman has yet another set of five stones. Using a simple two-pan balance she can determine the weight of any rock you give her weighing a whole number of pounds up to 242 pounds! What are the weights of her five stones?

(c) *Frogs*. Three frogs are placed on three vertices of a square. Every minute, one frog leaps over another frog, in such a way that the “leapee” is at the midpoint of the line segment whose endpoints are the starting and ending position of the “leaper.” Will a frog ever occupy the vertex of the square that was originally unoccupied?