

The Shape of Space - Part 2

Marin Math Circle

January 19 2011

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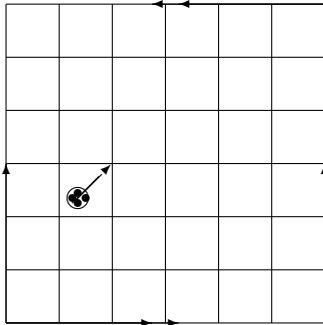
Most of this material is from *The Shape of Space* by Jeff Weeks. The word searches were taken from Torus Games at <http://geometrygames.org/TorusGames/index.html> . Download Torus Games for more word searches and many other games.

5 The Mobius band

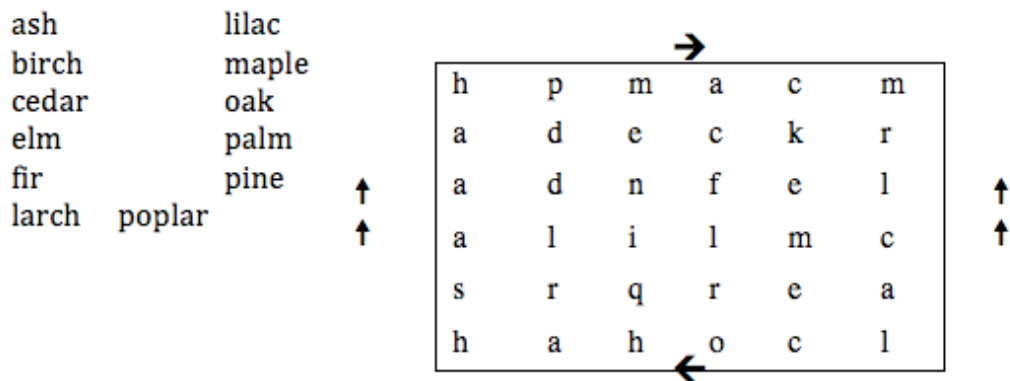
15. Cut a blank sheet of paper into four long strips. Make one strip into a cylinder by taping the ends with no twist, and make a second strip into a Mobius band by taping the ends together with a half twist (a twist through 180 degrees).
16. Mark an X somewhere on your cylinder. Starting at the X, draw a line down the center of the strip until you return to the starting point. Do the same for the Mobius band. What happens?
17. Make a gluing diagram for a cylinder by drawing a rectangle with arrows. Do the same for a Mobius band.
18. The gluing diagram you made defines a virtual Mobius band, which is a little different from a paper Mobius band. A paper Mobius band has a slight thickness and occupies a small volume; there is a small separation between its "two sides". The virtual Mobius band has zero thickness; it is truly 2-dimensional. Mark an X on your virtual Mobius band and trace down the centerline. You'll get back to your starting point after only one trip around!
19. A cylinder has two boundary circles How many boundary circles does a Mobius band have?
20. Take a pair of scissors and cut your paper Mobius band down its centerline. What do you get?
21. Take the result from the previous step and cut down its centerline. What do you get now?
22. Take another strip of paper and crease it to divide it into thirds lengthwise and then tape the ends to make a Mobius band.
 - (a) Predict what will happen when you cut your Mobius band along the creases.
 - (b) Cut along the pieces and see what happens.

6 The Klein Bottle

23. A ladybug on a Klein bottle walks in a straight line until she returns to her starting point. She walks 1 unit northward for every 1 unit eastward. Draw her path.



24. Try this word search on the Klein bottle. The arrows show how the sides are glued together. (Hint: on a Klein bottle, letters and their mirror images are not distinguishable, so “d” and “b” are the same, and so are “p” and “q”.) Remember that the top and the bottom are glued together and the left and right sides are glued together.



25. Play Tic Tac Toe on the Klein Bottle with a friend. Is it possible to get a Cats Game? Is there a winning strategy for the first player? How many essentially different first moves are there?
26. What do you get when you cut a Klein bottle in half? Does it depend on which way you cut it?

7 The 3-Torus

27. Suppose a 3-torus is made from a rectangular block 600 km by 800 km by 2400 km.
- (a) What is the shortest distance you can travel (in a straight line) to return to your starting place?

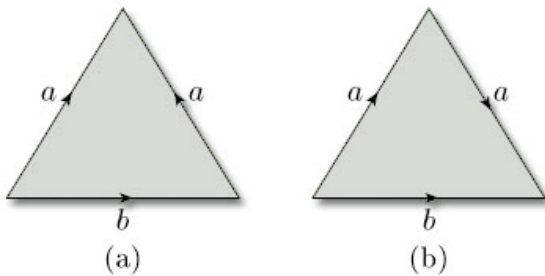
- (b) Suppose you want to construct two space stations as far away from each other as possible. Where should you put them? How far apart will they be?

8 Klein Space

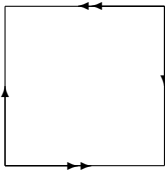
28. Find a 2-dimensional Klein bottle inside Klein space. Does it have one side or two sides?

9 More Gluing Diagrams

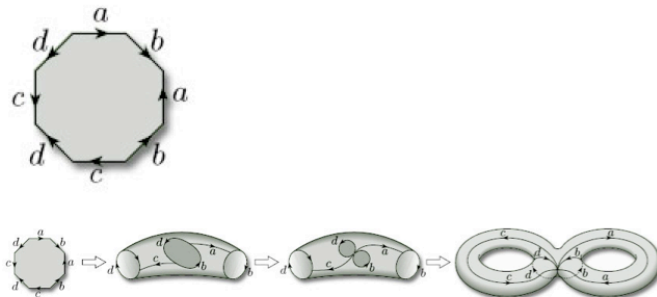
29. Which two surfaces are obtained by gluing the edges of each triangle as shown? You get two different surfaces, one for each triangle. Don't glue the two triangles together ... unless you really want to. Side b is not glued to anything.



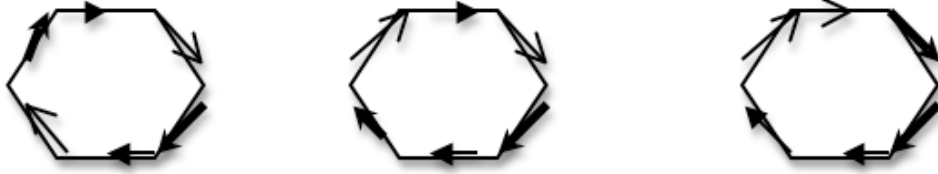
30. What do you get if you glue opposite sides of a square as drawn? Is it orientable?



31. An octahedron glued in the following pattern makes a 2-holed torus. Use the picture to convince yourself.

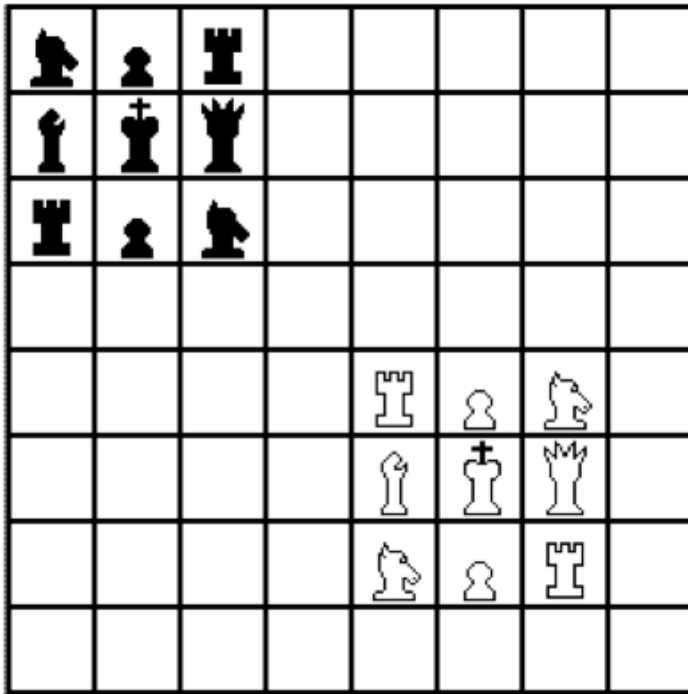


32. What surfaces do you get from each of the three hexagons with the following gluing patterns? On each hexagon, sides with matching arrows get glued together so that the arrow directions match. How can you tell from a gluing pattern if a surface is orientable?



10 For Chess Players

33. Find a friend and play a few games of chess on a torus. The usual starting position doesn't work (try it and you'll see why), but you can use the starting position below or make up a good one of your own. All pieces move normally, except the pawns, which can move one space in a straight line in any direction, and captures by moving one space diagonally in any direction.



34. When playing torus chess, if a bishop goes out the upper right-hand corner of the chessboard, where does he return?

35. In torus chess, is it possible for a knight and a bishop to simultaneously threaten each other?
36. Try chess on a Klein bottle. Use the starting position above, and glue the left and right side as you would for a torus, but glue the top and bottom with a flip.
37. If a bishop goes out the upper right-hand corner in Klein bottle chess, where does he return?
38. In Klein bottle chess, starting from the initial position above, can the white bishop capture the black rook in one move? Starting from the initial position, can a black knight capture a white rook in one move?