The Mad Veterinarian and Other Invariants

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"Invariants" sounds fancy but it really just means "Look for things that don't change."

1. Take any number. Add 10. Add 10 again. Add 10 again. Keep on adding. What stays the same? What changes in a steady direction?

2. What if you add 2 instead of 10? How about 9? What about other numbers?

3. What if you multiply by 10? Multiply by 9? Other numbers?

4. [Thanks to A. Bogolmony of http://cut-the-knot.com for reminding me of this idea] A mad veterinarian (that is, a mad scientist who studies animals) has invented an animal transmogrifying machine. If you put in two cats or two dogs, then one dog comes out of the machine. If you put in one cat and one dog, then one cat comes out. The veterinarian's goal is to end up with only one cat and no other animals. For example, you might start with three cats and a dog.
   What happens in this game? Can the veterinarian win?
   What if the veterinarian starts with a different collection of animals?

5. The veterinarian's old machine breaks. Now the veterinarian has dogs, cats, and mice. The new transmogrifying machine can take in any two *different* animals and then out comes the third animal. Can you win (by ending up with just one cat) if you start with three cats and a dog?

   What about other starting situations, like four of each animal for example?
   What about if you can reverse the rule when you wish, putting in *one* animal and having one of each of the other two animals come out?

6. What if the machine is changed so that when you put in two different animals, you then get *two* of the third animal? The goal is to get all the animals the same. What starting situations enable you to eventually win?

7. Now let's take a look at a machine with three functions. One converts a cat into two dogs and a mouse (or vice-versa): 1C <-> 2D 1M. A second machine does 1D <-> 1C 1M, and a third machine does 1M <-> 1C 3D. The general puzzle is to start with just one animal and replicate it: what's the fewest cats (more than one) that you can turn one cat into (with no mice or dogs left around)? Or, even more generally, starting with one cat can you describe all the combinations of animals you can end up with?

8. Here's a two-player game for a change: Player 1 writes a sequence of ten positive integers. Then player 2 writes a + or – sign in each of the nine spaces between the integers. In the end, if the final numeric result is odd, player 1 wins, and if even,

player 2 wins. Who should win this game, and how?

What if player 1 is given a bag with a certain collection of numbers, each of which can be used only once? For example, if they have a bag containing the numbers 1 through 12? 1 through 11? 1 through 10?

What if player 2 can use exactly one multiplication sign, and eight + or – signs, in the nine spaces?

What if player 2 gets exactly two multiplication signs (and 7 + or – signs)?

9. Another one-player game: Start with a stack of n boxes. At each move, as long as any stacks have more than one box, split one stack into two parts, say x boxes into y and z, and score yz points. How should you split them in order to maximize your score? What is the maximum score for each n?

10. Coin-flipping: Begin with some number of coins, say four for example, and set them on the table in a line, with a given starting sequence like HHTH for example. At each move, you may flip any two adjacent coins. You win if the final arrangement of the coins is all heads.

11. Coin-splitting: Begin with an infinite strip of squares, and a penny on one spot. At each move, you may either split the penny (remove it and put a penny on each adjacent spot) or merge two pennies (remove two pennies with exactly one space between them and put one on the space between; in other words, undoing the splitting operation). You may have any number of pennies on a given spot (but each move only splits one penny or merges two pennies into one). Starting with one penny, can you split and merge to end up with just one penny on the board in a different spot? What different spots are possible?

12. Cup flipping: A rotating platform has two cups on it which may be either up or down. You win if at any moment all the cups are up. On each turn, you can say "Flip North", "Flip South", or "Flip Both". How long does it take to guarantee a win?

    What if the table is rotated at random after each command?

    What if there are four cups instead of two? Other numbers?

    What if the cups have more than two positions?

13. Given some set of numbers (say, 2, 3, 5, 7) written on the board, erase any two of them (say, x and y) and write their sum, $x + y$. What happens in the long run?

14. What if, in the previous problem, $x + y$ is replaced by xy?

15. What if it is replaced with $xy + x + y$?

16. Two integers are written around a circle. At each step, replace each pair of adjacent numbers with the absolute value of their difference. What happens in the long run?

    What if you start with non-integers?

    What if you start with three numbers? 4 numbers?

17. Begin with a stack of chips. On your turn, you may take either 2 or 3 chips from the stack. If you take all the chips, you win. For what starting numbers of chips is there a winning strategy?

    What if you are only allowed to take 3 or 4 chips? What if you are only allowed to take 5 or 11 chips? (In any case, if there's a pile of chips left at the end, you lose, and if you take them all, you win).