

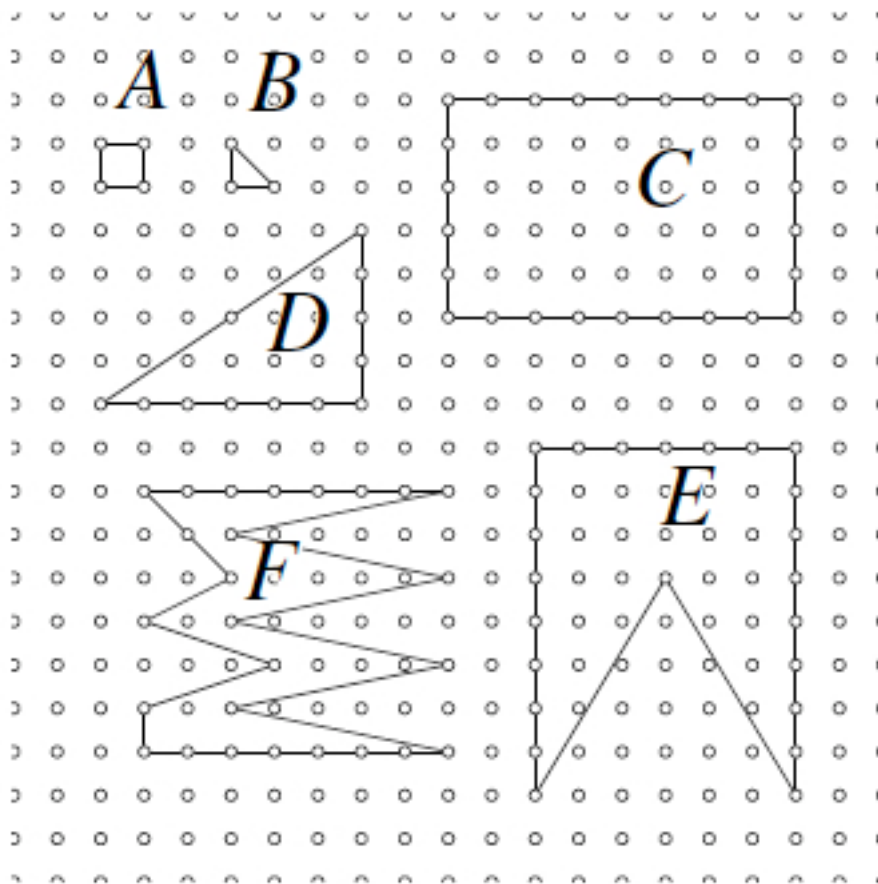
Pick's Theorem and Lattice Point Geometry

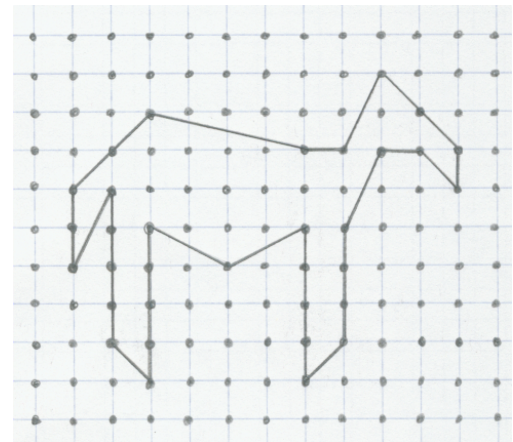
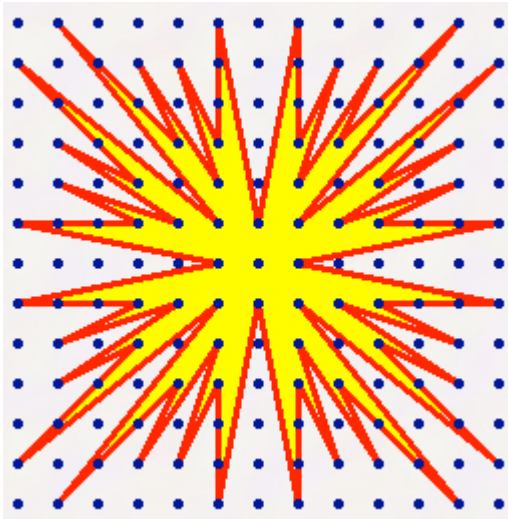
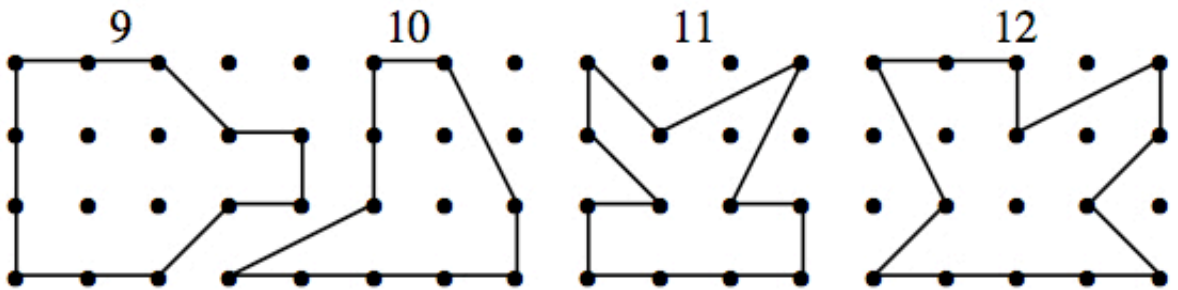
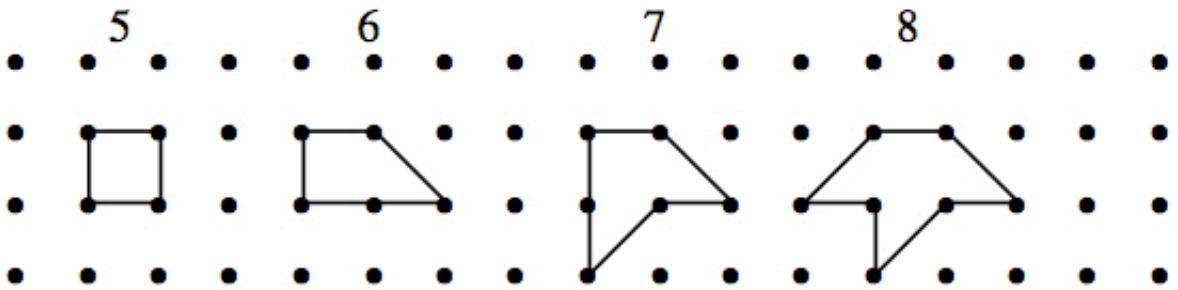
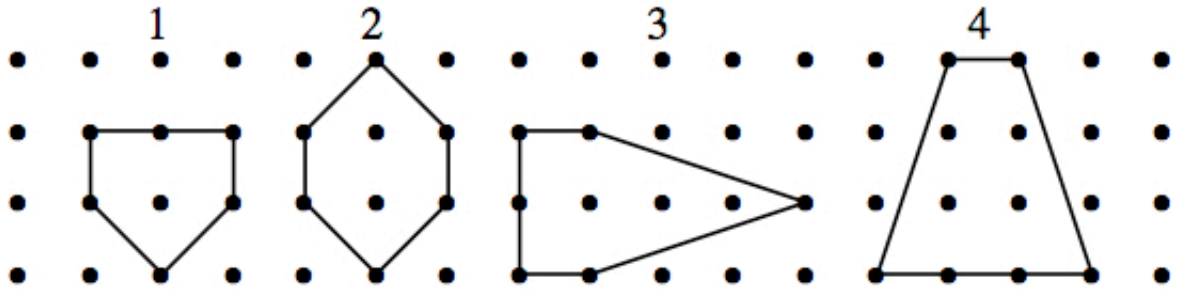
1 Lattice Polygon Area Calculations

Lattice points are points with integer coordinates in the x, y -plane. A *lattice line segment* is a line segment that has 2 distinct lattice points as endpoints, and a *lattice polygon* is a polygon whose sides are lattice line segments—this just means that the *vertices* of the polygon are lattice points. To begin, we'll consider *simple* polygons—these are polygons whose edges don't intersect (but later, we'll consider non-simple polygons as well). The polygons that we'll consider may be convex or concave.

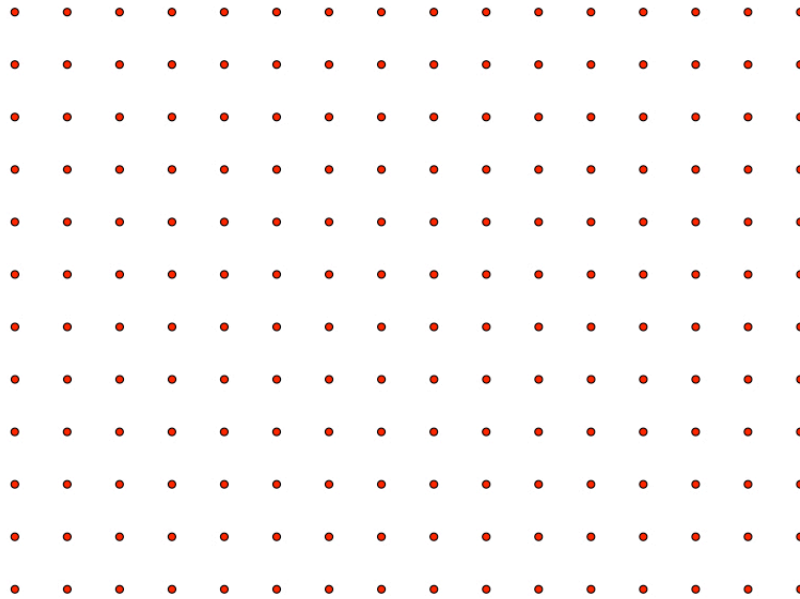
- Find the area of each of the following lattice polygons below and on the next page. Make a table that contains the following information for each polygon: the area of the polygon, the number of lattice points inside the polygon (I), and the number of lattice points on the boundary of the polygon (B). Can you make any observations or conjectures?

Polygon	Area	I	B	Polygon	Area	I	B	Polygon	Area	I	B	Polygon	Area	I	B
A				1				7				sunburst			
B				2				8				horse			
C				3				9							
D				4				10							
E				5				11							
F				6				12							

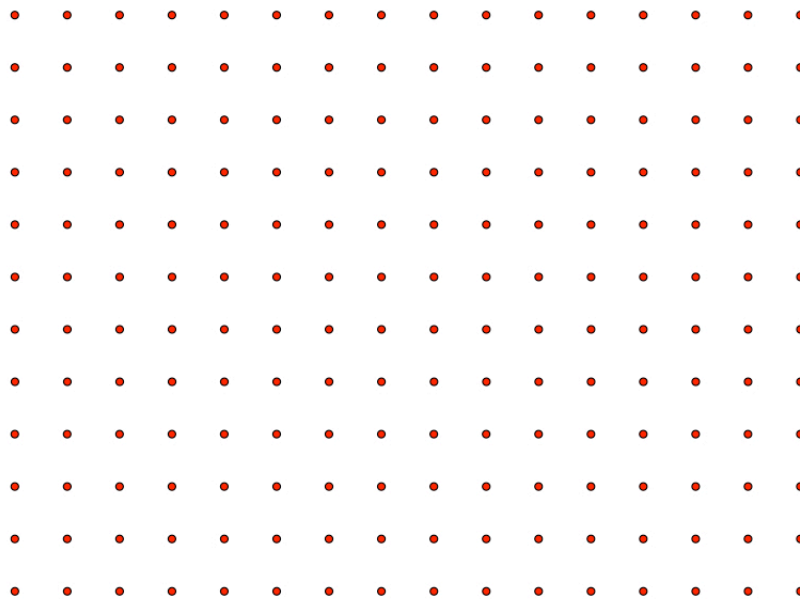




2. Construct at least 5 different polygons that contain 4 boundary lattice points and 6 interior lattice points. Keep in mind that the polygons do not need to be convex! What is the area of each polygon?



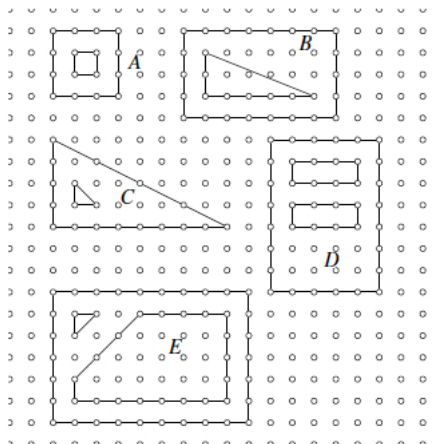
3. Let P be the triangle with vertices $(0, 0)$, $(3, 1)$, and $(1, 4)$. Find the area of P , the number of lattice points inside the polygon, and the number of lattice points on the boundary of the polygon.



4. Based on your work in problems 1-3, can you make a conjecture about a formula that relates the area of a lattice polygon to the number of lattice points inside the polygon and the number of lattice points on the boundary of the polygon? This formula is known as **Pick's Theorem**, and there are numerous beautiful proofs of this result. You can work through two of them using the outlines provided.

2 Problem-Solving with Lattice Geometry I

1. Is it possible to construct an equilateral lattice triangle? Why or why not?
2. Among all of the lattice points on a lattice line L through the origin $(0,0)$, there are exactly 2 that have minimum positive distance from the origin. Such lattice points are called *visible lattice points*. Show that a lattice point (a,b) is visible if and only if a and b are relatively prime.
3. Show that if (a,b) is a visible lattice point, then the lattice points on the line through (a,b) are all of the form $t(a,b)$, where t is an integer.
4. Find the equation of the line connecting 2 points $A(n,0)$ and $B(0,n)$, and show that this line contains all points of the form $(i,n-i)$, where i is an integer. There are $n-1$ such points between A and B . Connect each one of them with the origin $O(0,0)$. The lines divide $\triangle OAB$ into n small triangles. It is clear that the 2 triangles next to the axes (i.e. the triangle adjacent to the x -axis and the triangle adjacent to the y -axis) contain no lattice points in their interior. Prove that if n is prime, then each of the remaining triangles contains exactly the same number of interior lattice points. Find an expression (in terms of n) for the number of interior lattice points in each of these triangles.
5. (1987 IMO) Let n be an integer greater than or equal to 3. Prove that there is a set of n points in the plane such that the distance between any 2 points is irrational and each set of three points determines a non-degenerate triangle with rational area.
6. An $n \times n$ square is randomly tossed onto the plane. Prove that it may never contain more than $(n+1)^2$ lattice points.
7. Suppose that T is a lattice triangle and $I(T) = 0$. What are the possible values for $B(T)$? Can you find an example for each possible value?
8. If T is a lattice triangle with $I(T) = 1$, what are the possible values for $B(T)$? Can you find an example for each possible value?
9. Let L be a line in the plane, and suppose that the slope of L is irrational. Show that there is at most one lattice point on L . Give an example of a line with irrational slope containing one lattice point. Give an example of a line with irrational slope containing no lattice points.
10. Let L be a line with rational slope in the plane. Show that if there is a lattice point on L , then the y -intercept of L is rational. Show that if there is one lattice point on L , then there are infinitely many lattice points on L . Give an example of 2 lines with rational slope, one containing no lattice points, and the other containing infinitely many.
11. **Polygons with Holes.** In the following figure, there are 5 examples of polygons with holes. Polygons A, B, C have one hole, and polygons D and E have 2 holes. Find the area of each of these polygons. Make a table that contains the following information for each polygon: I, B , area, number of holes. Doing more examples if necessary, modify Pick's Theorem to derive a formula that works for polygons with holes. Then develop a proof of your conjecture.



Area for further study: more generally, investigate the relationship between Pick's Theorem and Euler characteristic.

12. Consider the lattice line segment from $(0, 0)$ to the point (a, b) , where a and b are any nonnegative integers. How many lattice points are there between $(0, 0)$ and (a, b) (excluding the endpoints)?
13. Let P be a lattice n -gon, with vertices $p_1 = (a_1, b_1)$, $p_2 = (a_2, b_2), \dots, p_n = (a_n, b_n)$. Let

$$d_i = \gcd(a_{i+1} - a_i, b_{i+1} - b_i).$$

Show that the number of lattice points on the boundary of P is

$$B(P) = \sum_{i=1}^n d_i.$$

14. Is it possible to construct an equilateral lattice square? A regular lattice hexagon? For which n is it possible to construct a regular lattice n -gon (i.e. a convex polygon that is equilateral and equiangular)?
15. (1993 Australian Mathematical Olympiad) The vertices of triangle ABC in the xy -plane have integer coordinates, and its sides do not contain any other points having integer coordinates. The interior of ABC contains only one point, G , that has integer coordinates. Prove that G is the centroid of ABC .

3 Problem-Solving with Lattice Geometry II

1. Prove that a lattice pentagon must contain a lattice point in its interior.
2. (Russian Mathematical Olympiad) If P is a convex polygon having the vertices on lattice points, prove that the smaller pentagon $S(P)$ determined by its diagonals contains, in its interior or on its boundary, a lattice point.
3. For which positive integers n is it possible to construct a lattice square with area n ?
4. This problem is an introduction to how Pick's Theorem generalizes in higher dimensions. First, we'll rewrite Pick's Theorem as follows. Let P be a lattice polygon, and let $L(P)$ denote the total number of lattice points in the interior and on the sides of P , so

$$L(P) = B(P) + I(P).$$

Then Pick's Theorem can be restated as follows:

$$L(P) = A(P) + \frac{1}{2}B(P) + 1.$$

The generalization of Pick's Theorem that we'll prove in this exercise describes how $L(P)$ changes as the polygon undergoes dilation by a positive integer. For each positive integer n , we define the lattice polygon nP as

$$nP = \{nx \mid x \in P\}.$$

Prove that

$$L(nP) = A(P)n^2 + \frac{1}{2}B(P)n + 1.$$

5. Let M be a bounded set in the plane with area greater than 1. Show that M must contain two distinct points (x_1, y_1) and (x_2, y_2) such that the point $(x_2 - x_1, y_2 - y_1)$ is an integer point (not necessarily in M).
6. Use the previous result to show that if S is a bounded, convex region in the plane that is symmetric about the origin and has area greater than 4, then S must contain an integer point other than $(0, 0)$.
7. Let $C(\sqrt{n})$ denote the circle with center $(0, 0)$ and radius \sqrt{n} .
 - (a) Find the number of lattice points on the boundary of $C(\sqrt{18})$, $C(\sqrt{19})$, $C(\sqrt{20})$, and $C(\sqrt{21})$.
 - (b) Can you find a general formula (or a rule for determining a formula, or any observations that might lead to a rule) for the number of lattice points on the boundary of $C(\sqrt{n})$?
8. Show that for every positive integer n , there is a circle in the plane that has exactly n lattice points on its circumference. This can be done in the following way:
 - (a) If $n = 2k$, show that the circle with center $(1/2, 0)$ and radius $\frac{1}{2} \cdot 5^{\frac{k-1}{2}}$ passes through exactly n lattice points.
 - (b) If $n = 2k + 1$, show that the circle with center $(1/3, 0)$ and radius $\frac{1}{3} \cdot 5^k$ passes through exactly n lattice points.
9. Let $(a, b) \in \mathbb{R}^2$. Suppose that for every positive integer n , there is a circle with center (a, b) containing exactly n lattice points. Show that at least one of a or b is irrational.
10.
 - (a) Find the number of lattice points in the interior and on the boundary of the circles $C(\sqrt{5})$, $C(\sqrt{7})$, and $C(\sqrt{10})$.
 - (b) (\star) Let $L(n)$ be the number of lattice points in the interior and on the boundary of the circle $C(\sqrt{n})$. Show that

$$\lim_{n \rightarrow \infty} \frac{L(n)}{n} = \pi.$$

Gauss used this result to approximate π .

11. (★) Use the previous result to show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Hint:

$$L(n) = 1 + 4 \sum_{0 < m \leq n} (d_1(m) - d_3(m)),$$

where $d_1(k)$ denotes the number of divisors of k congruent to 1 mod 4, and $d_3(k)$ denotes the number of divisors of k congruent to 3 mod 4.

12. (★) Let $V_{C(\sqrt{n})}$ be the number of visible lattice points in and on $C(\sqrt{n})$. Find a formula for

$$\frac{V_{C(\sqrt{n})}(n)}{L(n)},$$

and determine the limit

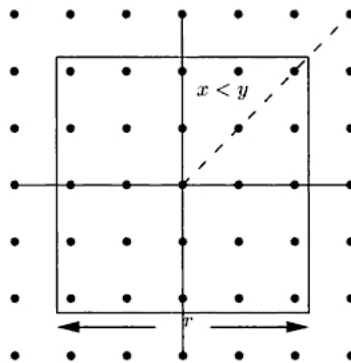
$$\lim_{n \rightarrow \infty} \frac{V_{C(\sqrt{n})}(n)}{L(n)}.$$

13. (★) Consider the square region $S(t)$ in the plane defined by the inequalities

$$|x| \leq t \text{ and } |y| \leq t,$$

where t is a positive real number. Let $N(t)$ denote the number of lattice points in this square, and let $V(t)$ denote the number of lattice points in the square that are *visible* from the origin. Show that

$$\lim_{t \rightarrow \infty} \frac{V(t)}{N(t)} = \frac{6}{\pi^2}.$$



14. Area for further study: do some research on Ehrhart's Theorem and the extension of Pick's Theorem to higher dimensions.