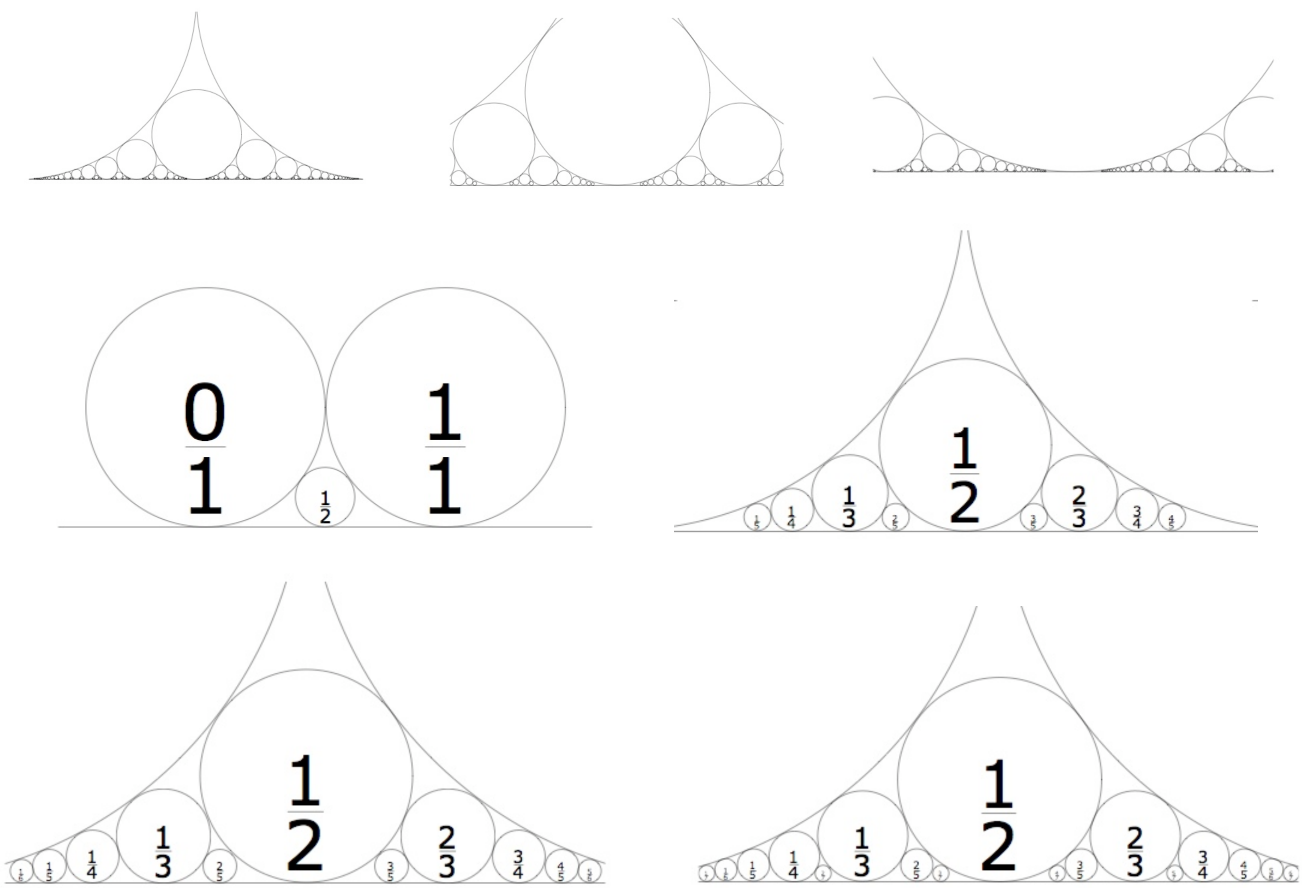
Farey Sequences and Ford Circles

Definition 1 Farey Sequence. The **Farey sequence of order** n, denoted F_n is the sequence of completely reduced fractions between 0 and 1 which, in lowest terms, have denominators less than or equal to n, arranged in order of increasing size.

$$\begin{aligned} F_1 &= \{0/1, 1/1\} \\ F_2 &= \{0/1, 1/2, 1/1\} \\ F_3 &= \{0/1, 1/3, 1/2, 2/3, 1/1\} \end{aligned}$$

Definition 2 Ford Circle. For every rational number p/q in lowest terms, the **Ford circle** $C(p, q)$ is the circle with center $(\frac{p}{q}, \frac{1}{2q^2})$ and radius $\frac{1}{2q^2}$. This means that $C(p, q)$ is the circle tangent to the x-axis at $x = p/q$ with radius $\frac{1}{2q^2}$. Observe that every small interval of the x-axis contains points of tangency of infinitely many Ford circles.



Problems.

1. Suppose that p_1/q_1 and p_2/q_2 are two successive terms of F_n. In this problem, we will use Pick's Theorem to prove that $p_2q_1 - p_1q_2 = 1$. Let T be the triangle with vertices $(0,0)$, (p_1,q_1), and (p_2,q_2).
   (a) Show that T has no lattice points in its interior, i.e. $I(T) = 0$.
   (b) Show that the only boundary points of T are the vertices of the triangle, i.e. $B(T) = 3$.
   (c) Conclude, using Pick's Theorem, that
   $$A(T) = \frac{1}{2}.$$
   (d) Use geometry to show that
   $$A(T) = \frac{1}{2}\left(p_2q_1 - p_1q_2\right).$$
   (e) Conclude that
   $$p_2q_1 - p_1q_2 = 1.$$
2. Prove that the representative Ford circles of two distinct fractions are either tangent at one point or wholly external. Moreover, the circles are tangent at one point precisely when the fractions are adjacent in some Farey sequence F_n.
3. Let a/b and a'/b' be the fractions immediately to the left and the right of the fraction $1/2$ in the Farey sequence of order n. Prove that b is the greatest odd integer less than or equal to n. Next, by experimenting with various choices of n, make and prove a conjecture about the value of $a + a'$.
4. Prove that the sum of the fractions in the Farey sequence F_n is equal to $\frac{1}{2}\left[1 + \sum_{j=1}^{n}\phi(j)\right]$.
5. Let a/b and a'/b' run through all pairs of adjacent fractions in the Farey sequence of order $n > 1$. Make and prove a conjecture about the values of
   $$\min\left(\frac{a'}{b'} - \frac{a}{b}\right) \text{ and } \max\left(\frac{a'}{b'} - \frac{a}{b}\right).$$
6. Consider the fractions from $0/1$ to $1/1$ inclusive in the Farey sequence of order n. Reading from left to right, let the denominators of these fractions be $b_1, b_2, \ldots, b_k$ so that $b_1 = b_k = 1$. By experimenting with various values of n, make and prove a conjecture about the value of $\sum_{j=1}^{k-1}\frac{1}{b_jb_{j+1}}$.
7. Suppose that $C(a,b)$ and $C(c,d)$ are tangent Ford circles. Prove that the largest Ford circle between them is $C(a+c, b+d)$, the Ford circle associated with the mediant fraction.
8. Suppose that a/b and c/d are adjacent terms in F_n (so that $C(a,b)$ and $C(c,d)$ are tangent Ford circles). Find a formula for all fractions that are adjacent to a/b in some Farey sequence.
9. Suppose that h_1/k_1, h_2/k_2, and h_3/k_3 are three consecutive terms in some Farey sequence F_n. Find the point of tangency of the circles $C(h_1,k_1)$ and $C(h_2,k_2)$, and the point of tangency of the circles $C(h_2,k_2)$ and $C(h_3,k_3)$
10. Investigate the relationship between the total area of Ford circles and the Riemann Hypothesis.